Center for Adaptive Optics
An NSF Science & Technology Center

Wave Optics

Don Gavel
UCO/Lick Observatory Laboratory for Adaptive Optics
University of California, Santa Cruz

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Course Objectives

- Gain a physical understanding of the wave and particle nature of light
- Acquire immediately useful information for use in the laboratory, for design studies, and theoretical development
- We will be light on the math, heavy on the concepts

Course Objectives (cont'd)

- Caveat
  - These note & lecture hopefully provide physical insight, but are not the definitive reference
- References
  - Anthony Siegman, **Lasers**, 1986 (Ch 16-20).
  - Web: **Eric Weisstein's World of Science** scienceworld.wolfram.com/physics/topics/Optics.html
  - Joseph Goodman, **Introduction to Fourier Optics**, 1996
  - Max Born and Emil Wolf, **Principles of Optics**, 7^{th} Ed., 2002.
  - George Reynolds, **The New physical optics notebook**, 1989.
  - Richard Feynman, **Lectures on Physics**, 1964.

Outline

- Light and photons
- Waves and interference
- Diffraction
- Fermat's principle, Marachal's condition, Lagrange Invariant
- Coherence
- Wave optics system modeling

The basics: what is light?

And God said

$$\nabla \cdot \mathbf{E} = 4\pi\rho$$

$$\nabla \cdot \mathbf{B} = 0$$

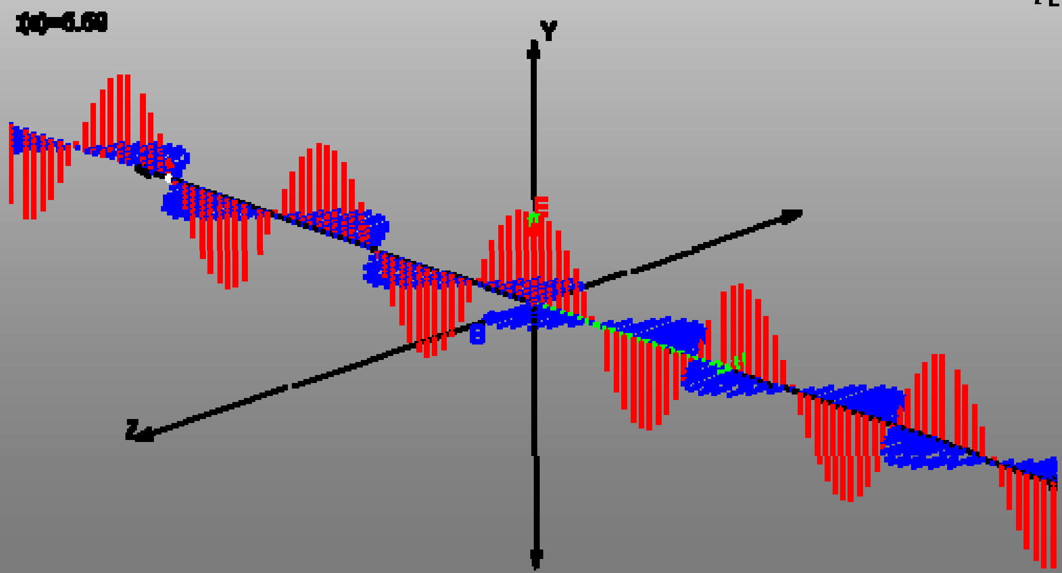
$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c}\mathbf{J}$$

…and there was light!

Light as EM wave

- Light is an electromagnetic wave phenomenon
- Waves propagate in free space according to the Helmholtz equation
- We detect its presence because the EM field interacts with the electron $\mathbf{F} = q[\mathbf{E} + (\mathbf{v}/c) \times \mathbf{B}]$



Helmholtz Equation

- In free space

$$\nabla^2 E = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} E$$

- Traveling waves

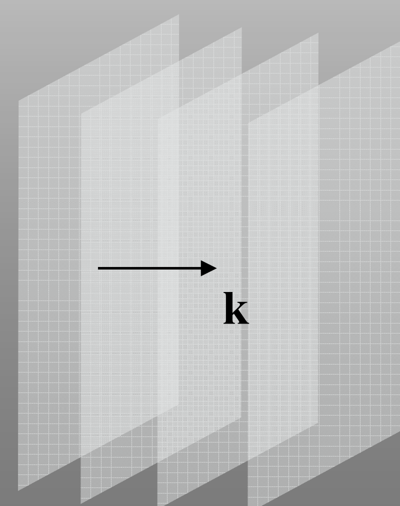
$$E(x,t) = E(0, t \pm x/c)$$

- Plane waves

$$E(\mathbf{x},t) = \tilde{E}(\mathbf{k}) e^{i(\omega t - \mathbf{k}\cdot\mathbf{x})}$$

Helmholtz Eqn.,
Fourier domain

$$k^2 E = (\omega/c)^2 E$$



Dispersion

- In free space

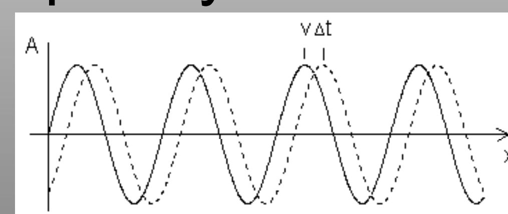
$$k = \omega / c$$ Dispersion relation is *linear*

$$k = 2\pi / \lambda$$ Wave number (**k** is wave vector)

$$\omega = 2\pi\nu$$ Wave frequency, ν

- In a medium
  - Plane waves have a *phase velocity*, and hence a wavelength, that depends on frequency

$$k(\omega) = \omega / v_{phase}$$



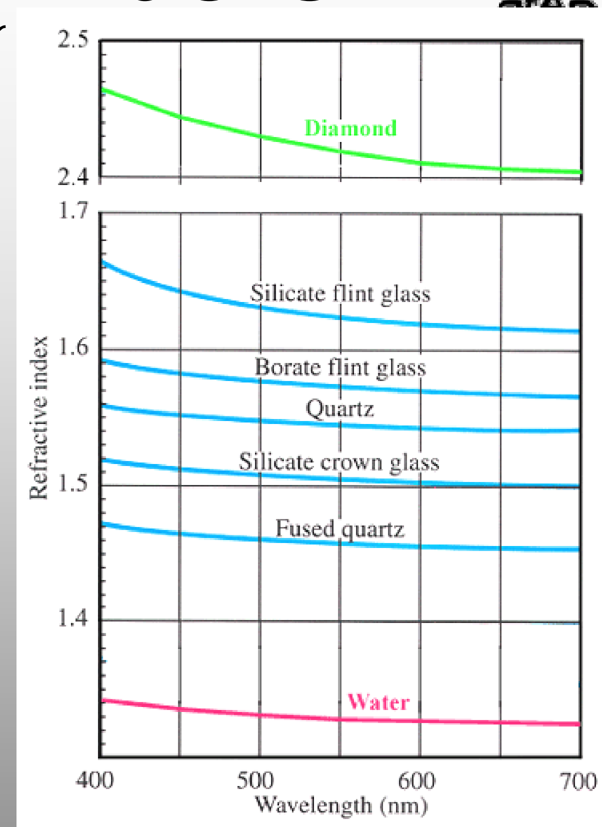
  - The “slow down” factor is the *index of refraction*, $n(\omega)$

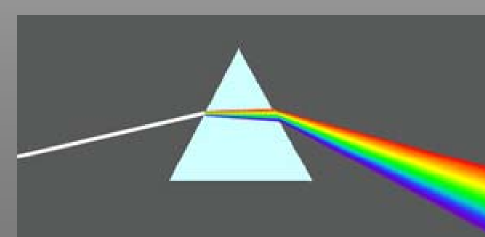
$$v_{phase} = c / n(\omega)$$

Some practical numbers

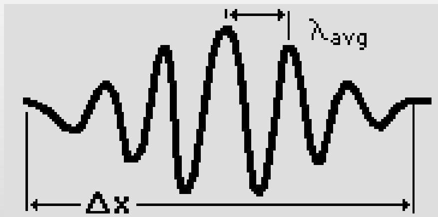
- Visible light
  - Wavelength λ = 0.4 - 0.75 μm
  - Velocity $c = 3 \times 10^8$ m/s
  - Frequency $\nu = 6 \times 10^{14}$ Hz
- Index of refraction
  - Air: 1.00029
  - Glass: ~1.5
  - Water: 1.33

Dispersion





Wave Packets and Group Velocity



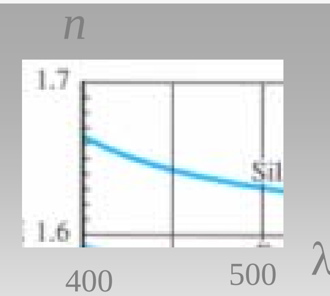
$$E(x,t) = \int \tilde{E}_k e^{i(\omega(k)t - kx)} \mathrm{d}k$$

$$= e^{i(\omega_0 t - k_0 x)} \int \tilde{E}_k e^{i(\Delta\omega t - \Delta k\, x)} \mathrm{d}\Delta k$$

Plane wave
phase velocity $v_{\mathrm{phase}} = \omega_0 / k_0$

Envelope
group velocity $v_{\mathrm{g}} = \Delta\omega / \Delta k$

<u>Example</u>:
Silica flint glass



$$v_g = \frac{\Delta\omega}{\Delta k} = \frac{\Delta(ck/n)}{\Delta k} = \frac{c}{n} - \frac{ck}{n^2}\frac{\Delta n}{\Delta k}$$

$$= \frac{c}{n}\left[1 - \frac{\Delta n/n}{\Delta k/k}\right] = \frac{c}{n}\left[1 + \frac{\Delta n/n}{\Delta\lambda/\lambda}\right]$$

λ = 400-500nm
n = 1.63-1.66

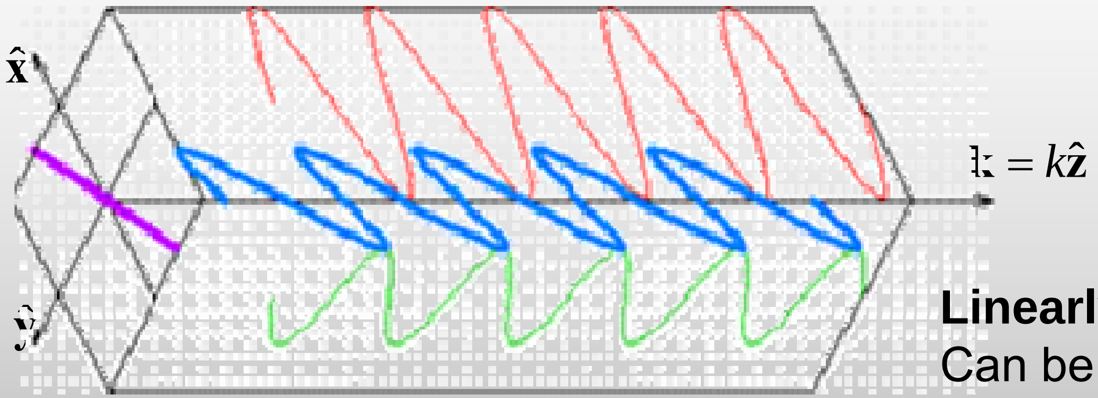
$$\frac{\Delta\lambda}{\lambda} = \frac{100}{450} = 0.222$$

$$\frac{\Delta n}{n} = \frac{-0.03}{1.645} = -0.207$$

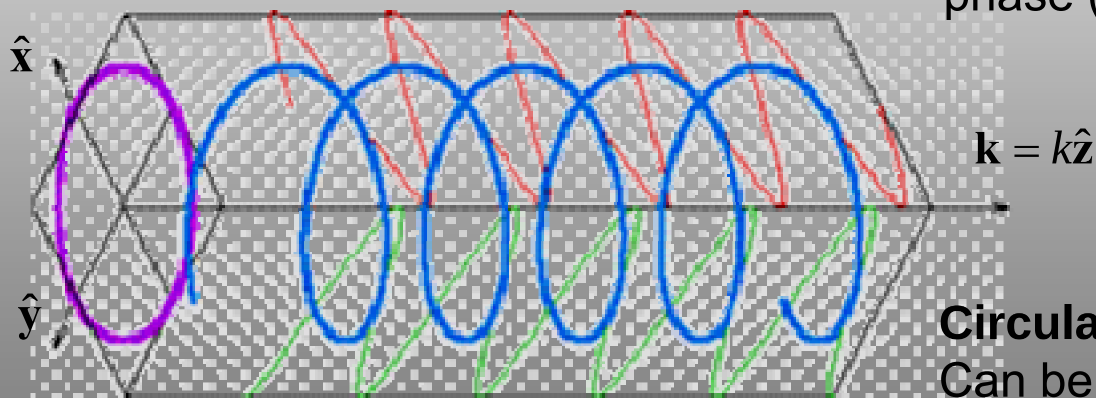
$$v_g = 0.918 v_{phase}$$

$$v_{phase} = \frac{c}{n} = 0.61c$$

Polarization



Linearly Polarized
Can be written as a sum of two linearly polarized waves, in phase (x and y components)



Circularly Polarized
Can be written as a sum of two linearly polarized waves, $\pi/2$ radians out of phase.

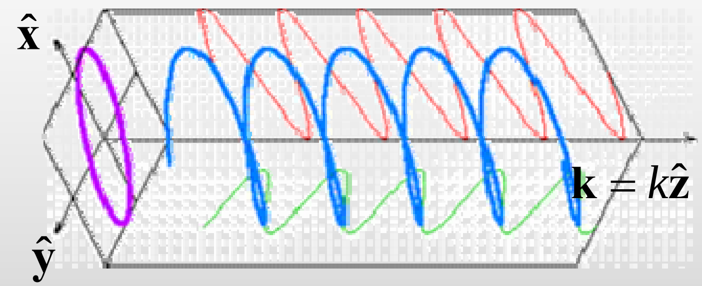
Polarization – Stokes parameters

- 4 parameters of polarization
  - Total intensity
  - Rotation of ellipse
  - Ratio of major to minor axis of ellipse
  - Degree of polarization
- $$\mathbf{E} = \left(\hat{\mathbf{x}}E_x + \hat{\mathbf{y}}E_y\right)e^{i(\omega t - kz)} \qquad E_y = \left|E_y\right|e^{i\delta}$$

$$I = \left\langle E_x E_x^* + E_y E_y^* \right\rangle \quad Q = \left\langle E_x E_x^* - E_y E_y^* \right\rangle$$

$$U = \left\langle E_x E_y^* + E_y E_x^* \right\rangle \quad V = i\left\langle E_x E_y^* - E_y E_x^* \right\rangle$$

- Waves can be a sum of mixed polarization waves ("quasi-monochromatic")



- Degree of polarization

$$\Pi = \sqrt{Q^2 + U^2 + V^2} \Big/ I$$

- Degree of linear polarization

$$\Pi_L = \sqrt{Q^2 + U^2} \Big/ I$$

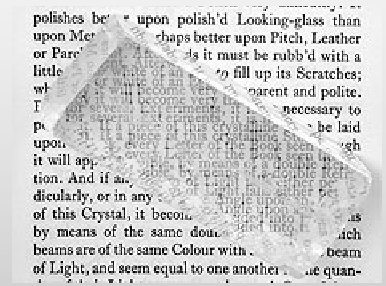
- Degree of circular polarization

$$\Pi_C = V / I$$

http://scienceworld.wolfram.com/physics/StokesParameters.html

Birefringence

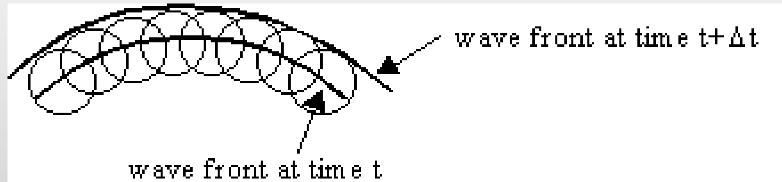
- Medium can have different index of refraction for each component of polarization



- Polarization splitter (Wallaston Prism)
- Waveplates
  - Linear to circular polarization (λ/4 plate)
  - Rotate linear polarization (λ/2 plate)

Optical path – Fermat's principle

- Huygens' wavelets
- Optical distance to radiator



$$\Delta x = c\Delta t / n$$

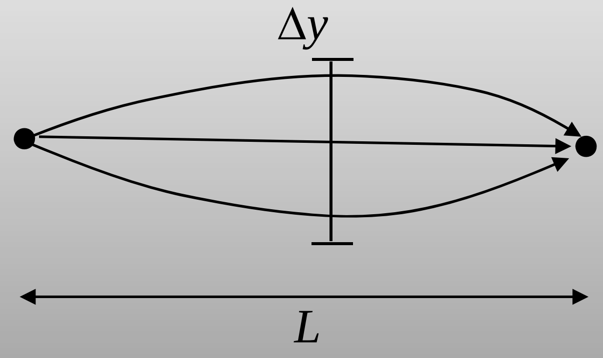
$$OPD = \int n \mathrm{d}x$$

$$\phi = \omega \Delta t = k \times OPD$$

- Wavefronts are iso-OPD surfaces
- Light ray paths are paths of least* time (least* OPD)

*in a local minimum sense

Why doesn't light prefer other paths?



- Waves arriving in phase add
- Waves arriving out of phase cancel

$$OPD_0 = L$$

$$OPD_\Delta = L + \frac{1}{2}\frac{\Delta y^2}{L}$$

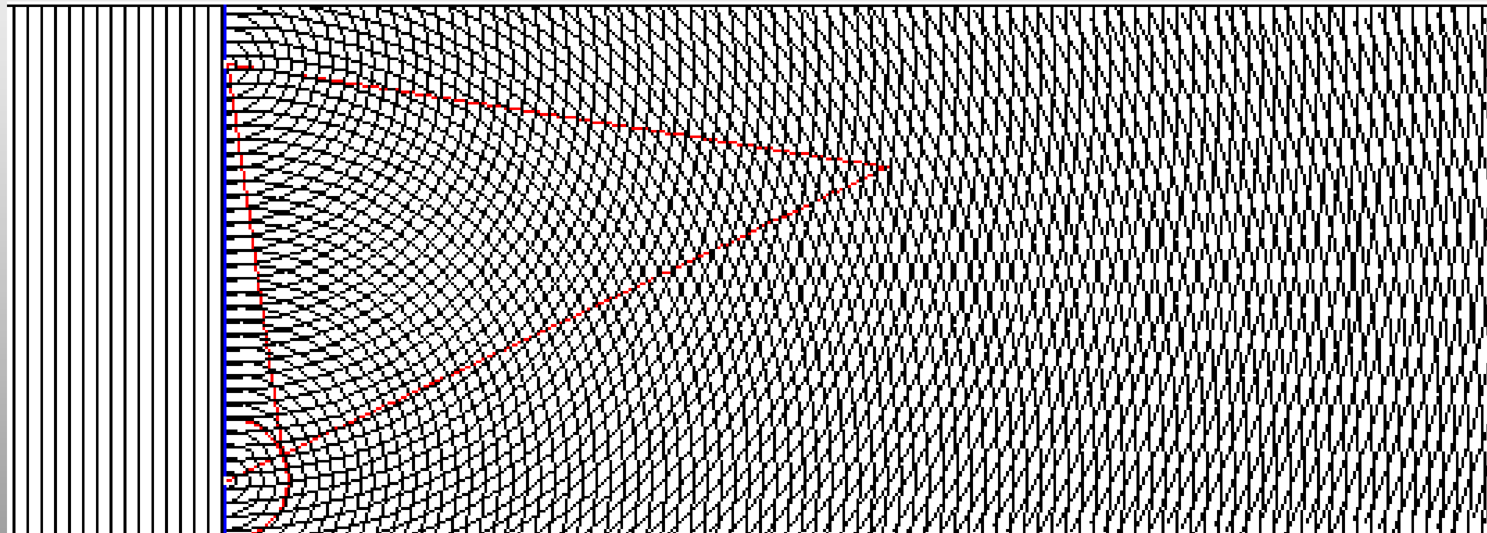
$$OPD_\Delta - OPD_0 < \lambda/2 \Rightarrow \Delta y < \sqrt{\lambda L}$$

Fresnel zone

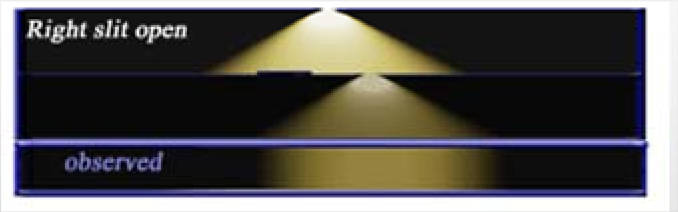
Light as particles

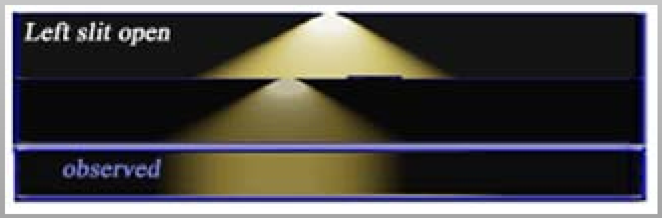
- Light originates as photons each emitted by the oscillation of a single atom.
- Light travels as a wave via all possible paths (paths of ‘least time’).
- Light, when detected, is realized as single-photon events distributed according to the intensity of the wave.

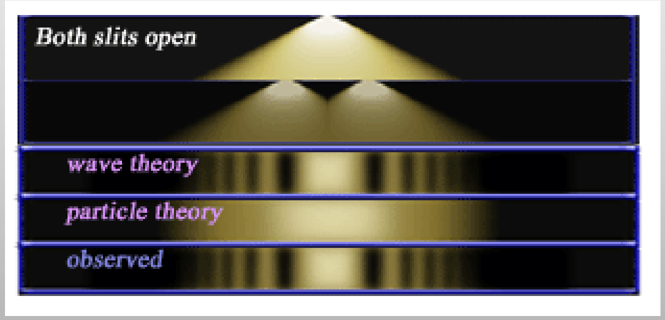
Detecting photons – Young's double slit experiment

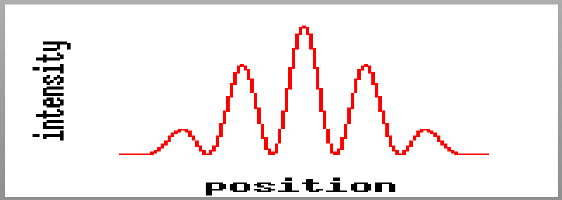


Young's Two-Slit Experiment



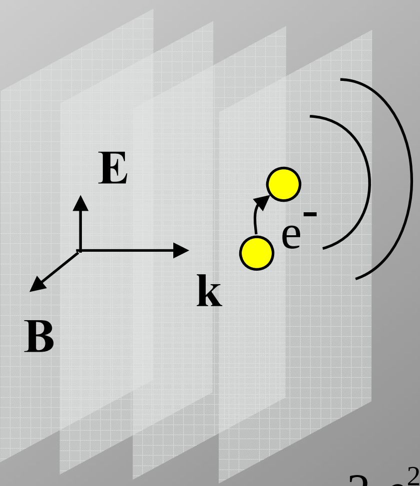




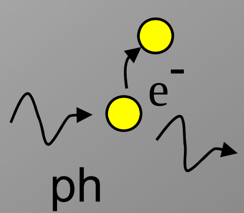


Compton scattering

- Photons interact only with charged particles (electrons, in ordinary life)
- Classical: The EM wave exerts force on the electron. The electron moves and thus emits an EM wave.
- Quantum: A photon collides with an electron and exchanges energy and momentum with it. The photon scatters with a different wavelength.



$$m\ddot{x} = q\left[\mathbf{E} + \left(\mathbf{v}/c\right) \times \mathbf{B}\right] - \frac{2}{3}\frac{e^2}{c^3}\dddot{x}$$



$$\lambda' - \lambda = \frac{2h}{mc}\sin^2\frac{\theta}{2}$$

Light has energy and momentum

- Classical: electromagnetic field energy and momentum density

$$E_{EM} = \frac{1}{8\pi}\int \left(E^2 + B^2\right) \mathrm{d}V$$

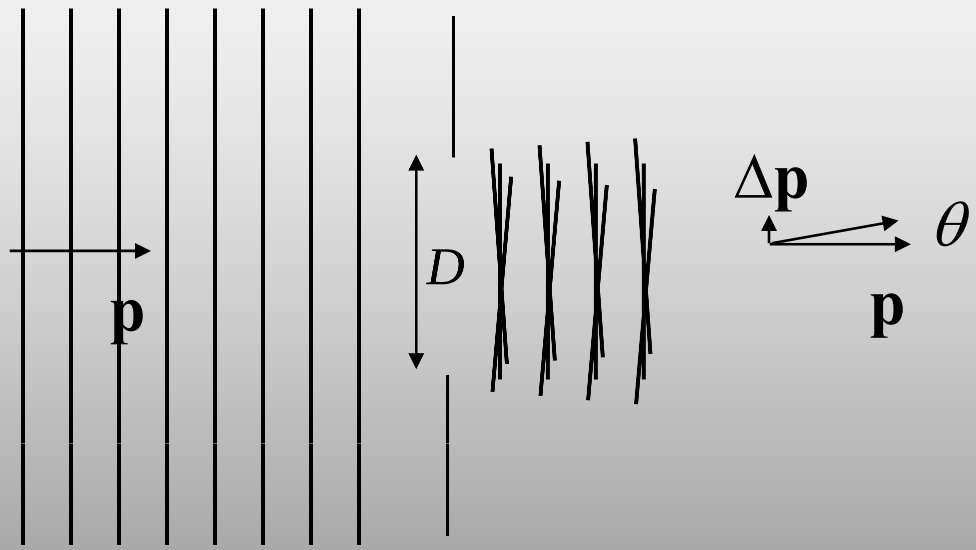
$$\mathbf{p}_{EM} = \frac{E_{EM}}{c}\hat{\mathbf{k}} = \frac{1}{4\pi c}\int \left(\mathbf{E}\times\mathbf{B}\right) \mathrm{d}V$$

- Quantum: photon energy and momentum

$$E_{ph} = h\nu$$

$$\mathbf{p}_{ph} = \frac{E_{ph}}{c}\hat{\mathbf{k}} = \hbar\mathbf{k}$$

Diffraction as a particle phenomenon



Uncertainty principle

$$\Delta x \Delta p \cong h$$

Photon momentum

$$p = h/\lambda$$

$$\Delta \mathbf{x} = \infty$$

$$\Delta \mathbf{p} = 0$$

$$\Delta x = D$$

$$\Delta p = h/D$$

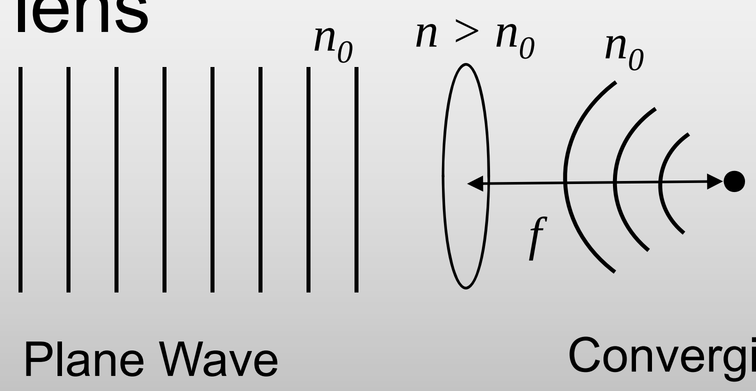
$$\Rightarrow$$

Law of diffraction

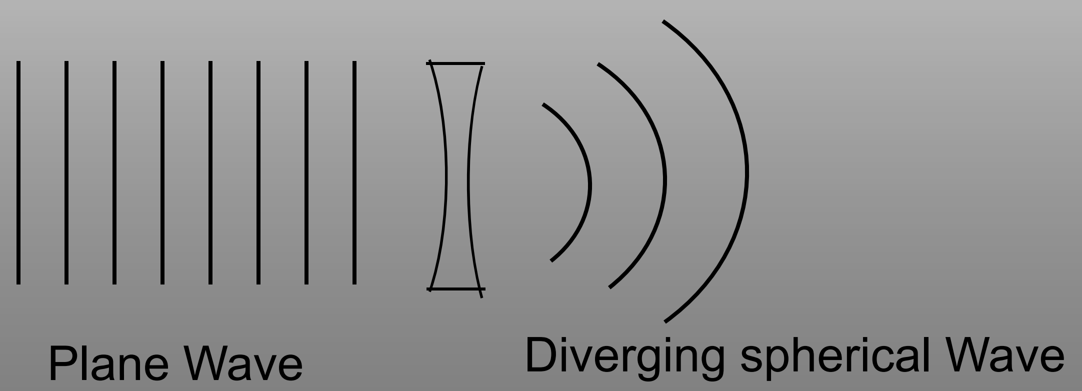
$$\theta \cong \Delta p/p = \lambda/D$$

Ideal lens

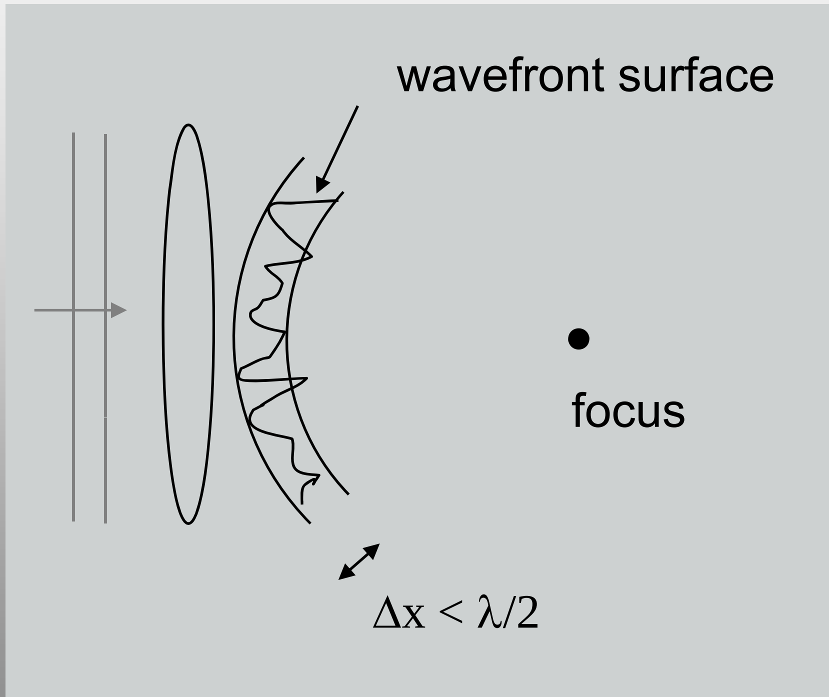
- Positive lens



- Negative lens



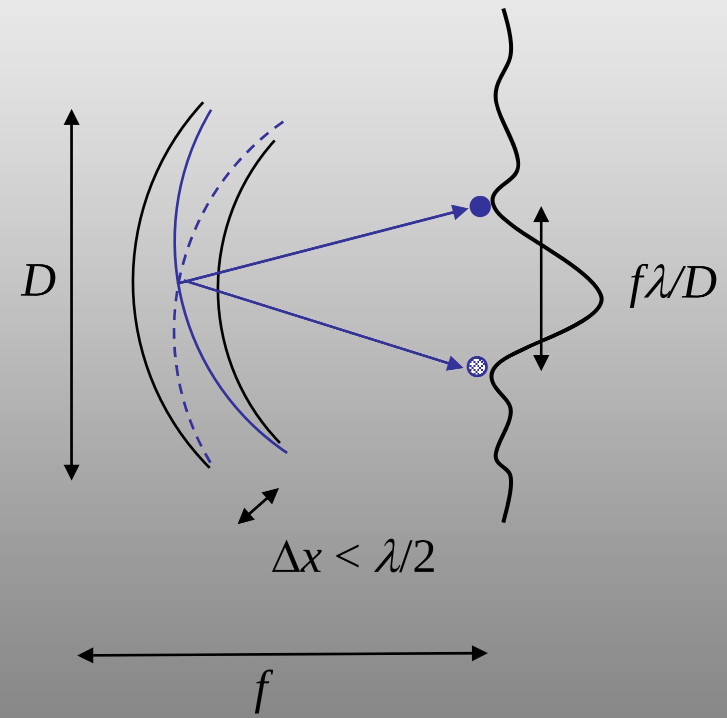
Marechal's condition



- If wavefront phase is contained within confocal spheres λ/2 apart everywhere where the intensity is significant
- The waves will add up at the focus
- Consequence of Fermat's principle

Diffraction angle

- Tip/Tilt allowed by Marechal's condition

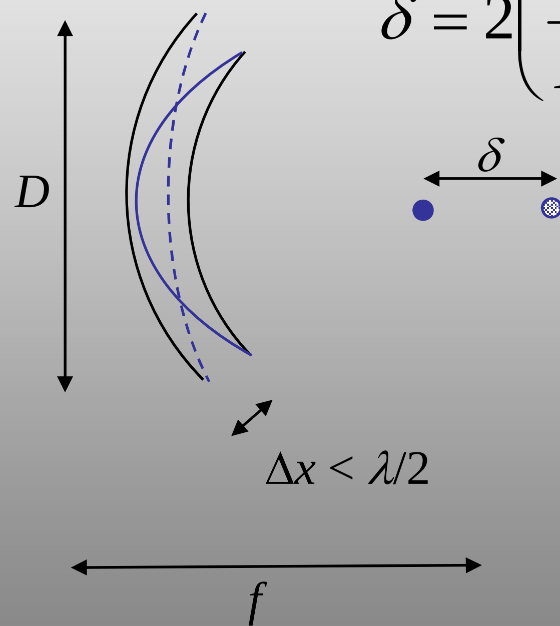


$$\theta = \frac{\lambda}{D}$$

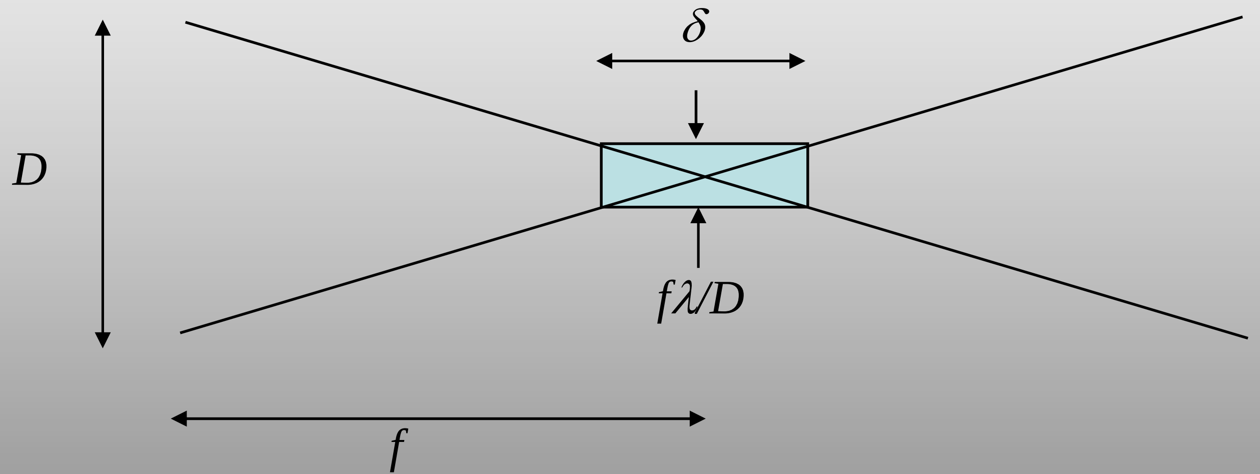
Depth of Focus

- Defocus allowed by Marechal's condition

$$\delta = 2\left(\frac{f}{D}\right)^2 \lambda = 2\left(f\,\#\right)^2 \lambda$$

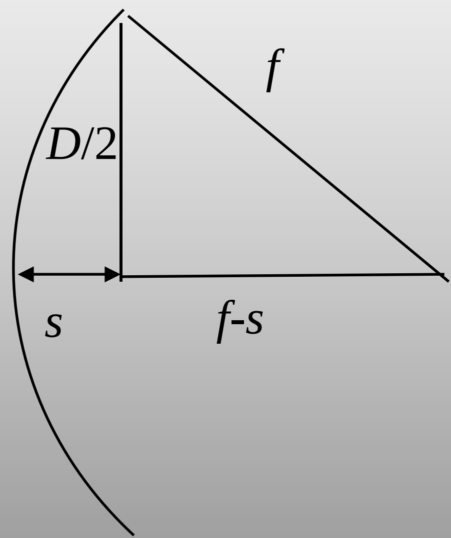


Another derivation of depth of focus



$$\frac{f}{D} = \frac{\delta/2}{f\,\lambda/D} \Rightarrow \delta = 2\left(\frac{f}{D}\right)^2 \lambda$$

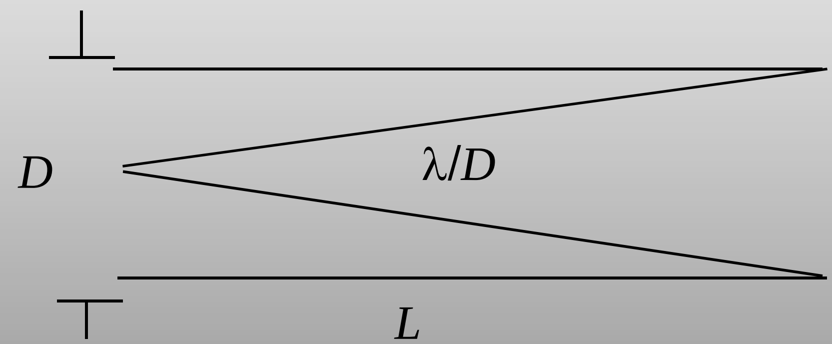
Wavefront sag



$$f^2 = (f - s)^2 + (D/2)^2$$
$$\cong f^2 - 2fs + D^4/4$$
$$s = \frac{D^2}{8f}$$

Rayleigh range

- Distance where diffraction overcomes paraxial beam propagation

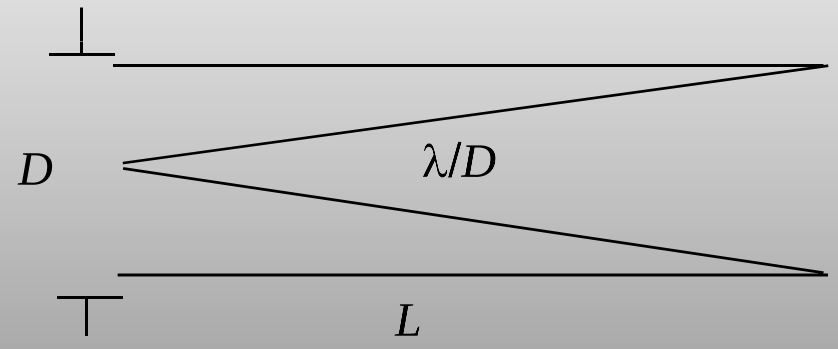


$$\frac{L\lambda}{D} = D \Rightarrow L = \frac{D^2}{\lambda}$$

- Also: wavefront sag is less than half a wave

Fresnel number

- Number of Fresnel zones across the beam diameter



$$N = \frac{D}{L\lambda/D} = \frac{D^2}{L\lambda}$$

Paraxial beams

- Helmholtz equation

$$\nabla^2 E = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}E$$

$$E(x,y,z,t) = u(x,y,z)e^{i(-kz)}e^{i\omega t}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} - 2ik\frac{\partial u}{\partial z} - k^2 u = -\frac{\omega^2}{c^2}u \quad , \qquad k = \frac{\omega}{c}$$

- Paraxial approximation

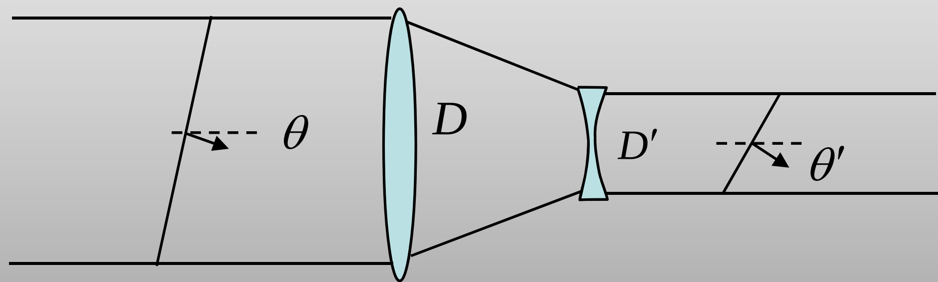
$$\left|\frac{\partial^2 u}{\partial z^2}\right| << \left|2k\frac{\partial u}{\partial z}\right| \text{ or } \left|\frac{\partial^2 u}{\partial x^2}\right| \text{ or } \left|\frac{\partial^2 u}{\partial y^2}\right|$$

- Paraxial wave equation

$$\nabla_t^2 u - 2ik\frac{\partial u}{\partial z} = 0$$

Lagrange invariant

- If the beam diameter is condensed, the angles increase proportionally

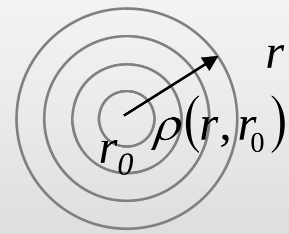


$$\Xi = \theta D = \theta' D'$$

- Conservation of energy
  – Flux is proportional to Ξ^2

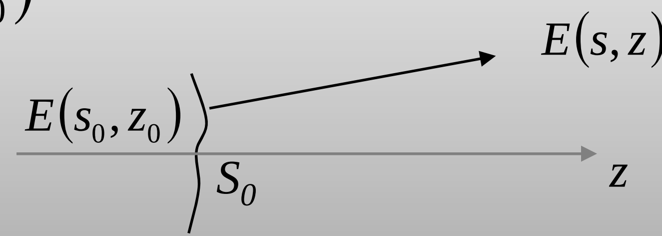
Huygens' Integral

- Wavelet



$$E(r;r_0) = \frac{e^{-ik\rho(r,r_0)}}{\rho(r,r_0)}$$

- Huygens' integral



$$E(s,z) = \frac{i}{\lambda} \iint_{S_0} E(s_0, z_0) \frac{e^{-ik\rho(r,r_0)}}{\rho(r,r_0)} \cos\theta(r,r_0) \mathrm{d}S_0$$

Fresnel approximation

- Spherical wavelet approximated as paraboloid
- Useful for computer numerical wave-optic propagation (it's a convolution)

$$u(x,y,z) = \frac{i}{L\lambda}\iint u_0(x_0,y_0,z_0)\exp\left[-i\pi\frac{(x-x_0)^2+(y-y_0)^2}{L\lambda}\right]\mathrm{d}x_0\mathrm{d}y_0$$

- Fourier domain: plane wave components

$$\mathbf{k} = (k_x, k_y, k_z)$$
$$k_\perp = k(\sin\theta_x, \sin\theta_y)$$
$$k \cong k_z$$

$$\tilde{u}(k_\perp, L) = \tilde{u}(k_\perp, 0)\times\exp\left[i\lambda k_\perp^2 L\right]\times e^{-ikL}$$

An alternative computational formula

- Fresnel approx to Huygens' integral (again)

$$u(x,y,z)=\frac{i}{L\lambda}\iint u_0(x_0,y_0,z_0)\exp\left[-i\pi\frac{(x-x_0)^2+(y-y_0)^2}{L\lambda}\right]\mathrm{d}x_0\mathrm{d}y_0$$

- Alternative Fourier transform formulation for numerical propagation

$$u(x,L)=\exp\left(\frac{i\pi x^2}{L\lambda}\right)\sqrt{\frac{i}{L\lambda}}\iint u'(x_0,0)\times\exp\left[i\frac{2\pi}{L\lambda}xx_0\right]\mathrm{d}x_0$$

$$u'(x_0,0)=u_0(x_0,z_0)\exp\left(-i\frac{\pi x_0^2}{L\lambda}\right)$$

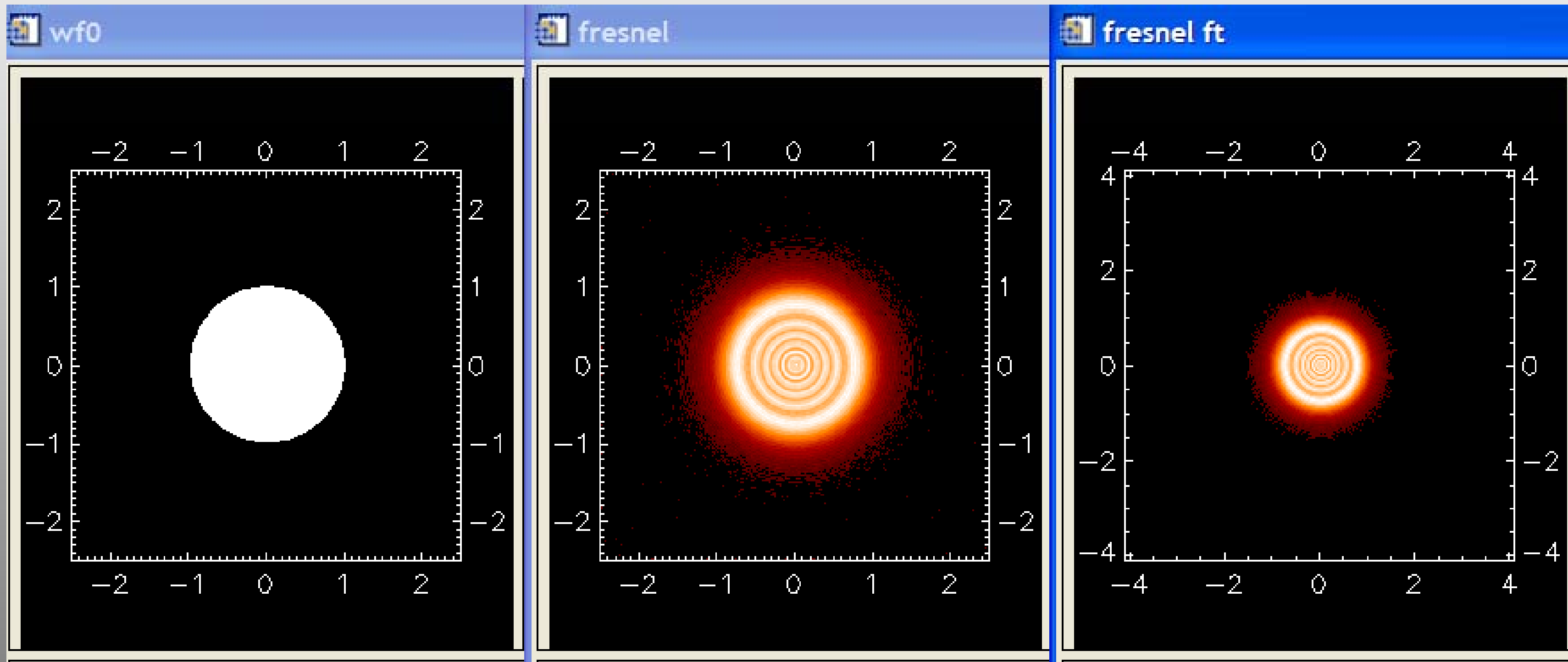
Fresnel propagation in IDL

```
;+
; fresnel.pro - fresnel propagation
;
;    Implement's Fresnel's approximation to Huygens' integ
;    for a complex wavefront propagating paraxially from a
;    z=0 to a plane at z=L.
;    Refernce: Siegman, Ch 16, eqn (79)
;    translated from fresnel.vm
;
; USAGE:
;  wfL = fresnel(wf0,du,L,lambda)
;
; INPUTS:
;  wf0 - complex wavefront at a given optical plane
;  du - spacing on the wavefront grid, in meters
;  L - real length of propagation, in meters
;  lambda - real wavelength of light, in meters
;
; OUTPUT:
;  wfL - complex wavefront at distance L
;
; ALGORITHM:
;   ~wfL = ~wf0 * exp{ + i k_perp^2 * L / 2*k }
;   where ~ indicates Fourier transform
;-
```

```
function fresnel,wf0,du,l,lambda
  k = 2*!pi/lambda
  n = (size(wf0))(1)
  df = 1./(n*du)
  dk = 2.*!pi*df
  fwf0 = shift(fft(shift(wf0,n/2,n/2)),n/2,n/2)
  k0 = -(n/2)*dk
  kf = (n/2-1)*dk
  r = findgen(n)*dk + k0
  kx = transpose(r) ## make_array(n,1,/float,value=1)
  ky = transpose(kx)
  kperp2 = kx*kx + ky*ky
  propPhase = kperp2*L/(2*k)
  propMag = 0*kperp2 + 1
  i = complex(0,1)
  prop = propMag*Exp(i*propPhase)
  fwfl = fwf0*prop
  wfl = shift(fft(shift(fwfl,n/2,n/2),/inverse),n/2,n/2)
  return,wfl
end
```

Plane wave propagation from a circular aperture

L = 2% of Rayleigh range



Far field

- Each tilted plane wave produces a point image, at angle θ
- Sum of plane waves → sum of points = Fourier transform

$$u(\theta_x, \theta_y, \infty) \propto \iint u_0(x, y, 0) \exp(ik\boldsymbol{\theta} \cdot \mathbf{x}) \mathrm{d}x \mathrm{d}y$$

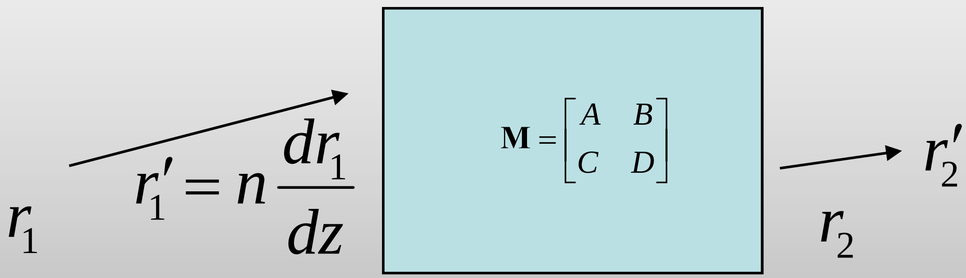
- This is just a variation of the alternative form, with $L \to \infty$

When to use each method

- Fresnel approximation
  - $L <$ ~10% of Rayleigh range
- Fresnel approximation alternative form
  - $L >$ ~10% of Rayleigh range
- Far field
  - $L = \infty$

ABCD ray optics

- Optics systems as 2x2 linear ray transformations

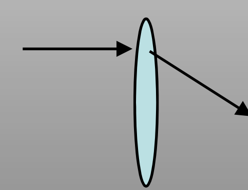


- Free space propagation

$$\begin{bmatrix} 1 & L/n \\ 0 & 1 \end{bmatrix}$$

- Thin lens

$$\begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix}$$



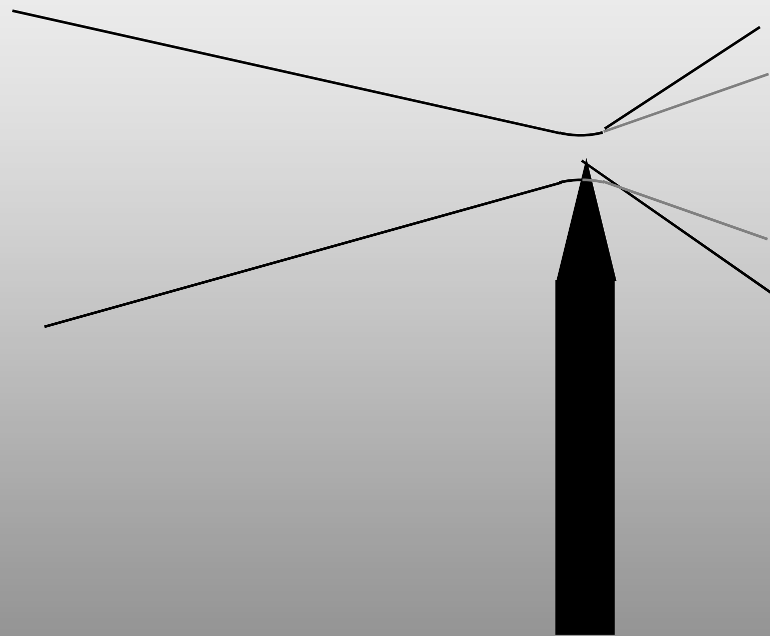
- Curved mirror
- Dielectric interface
- Transversely graded index
- …

Huygens' integral (Fresnel approximation) through ABCD system

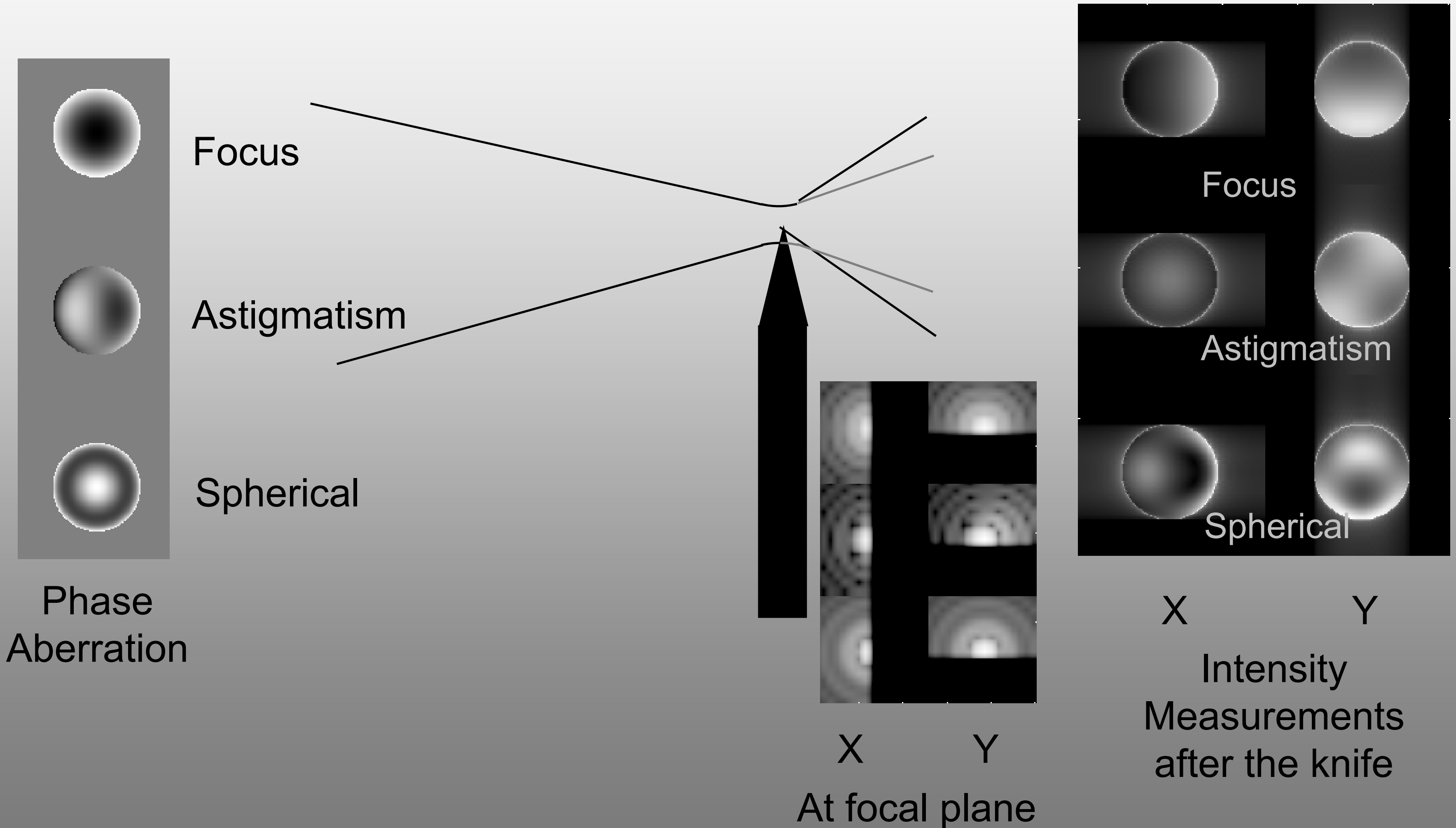
$$u_2(x_2) = e^{-ikL_0} \iint K(x_2, x_1) u_1(x_1) \mathrm{d}x_1$$

$$K(x_2, x_1) = \sqrt{\frac{i}{B\lambda_0}} \exp\left[-i \frac{\pi}{B\lambda_0} \left(A x_1^2 - 2 x_1 x_2 + D x_2^2 \right) \right]$$

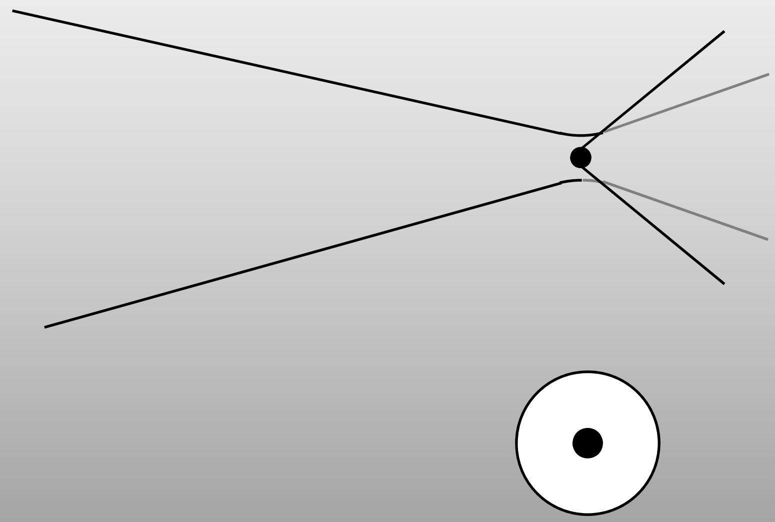
Example: Knife Edge Test



Example: Knife Edge Test

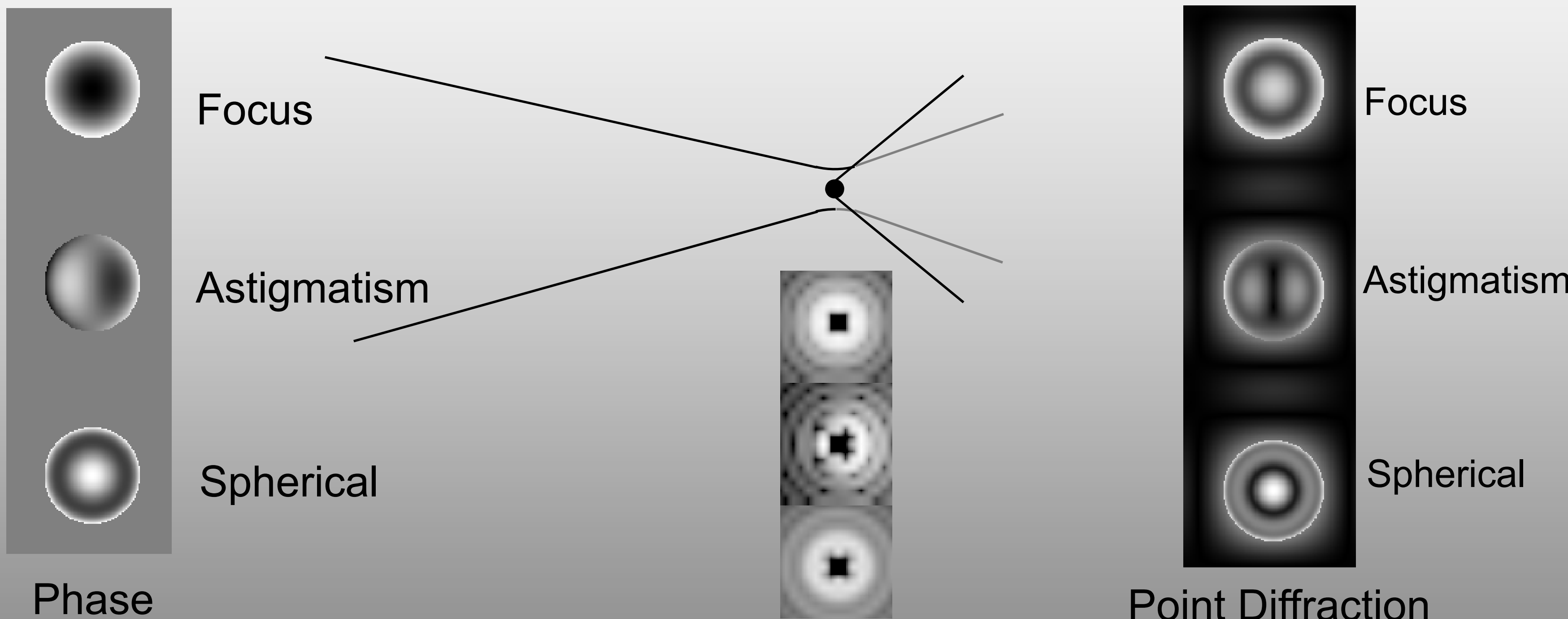


Example: Point Diffraction



- Focal Plane Block at center of beam approximately 1/2 diffraction limit

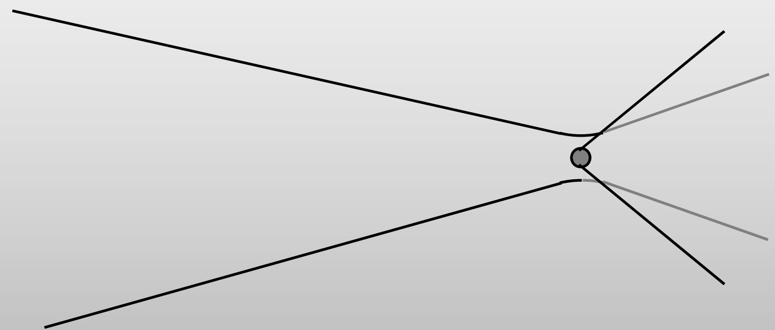
Example: Point Diffraction

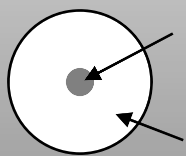


Phase Aberration

Point Diffraction Intensity Measurements

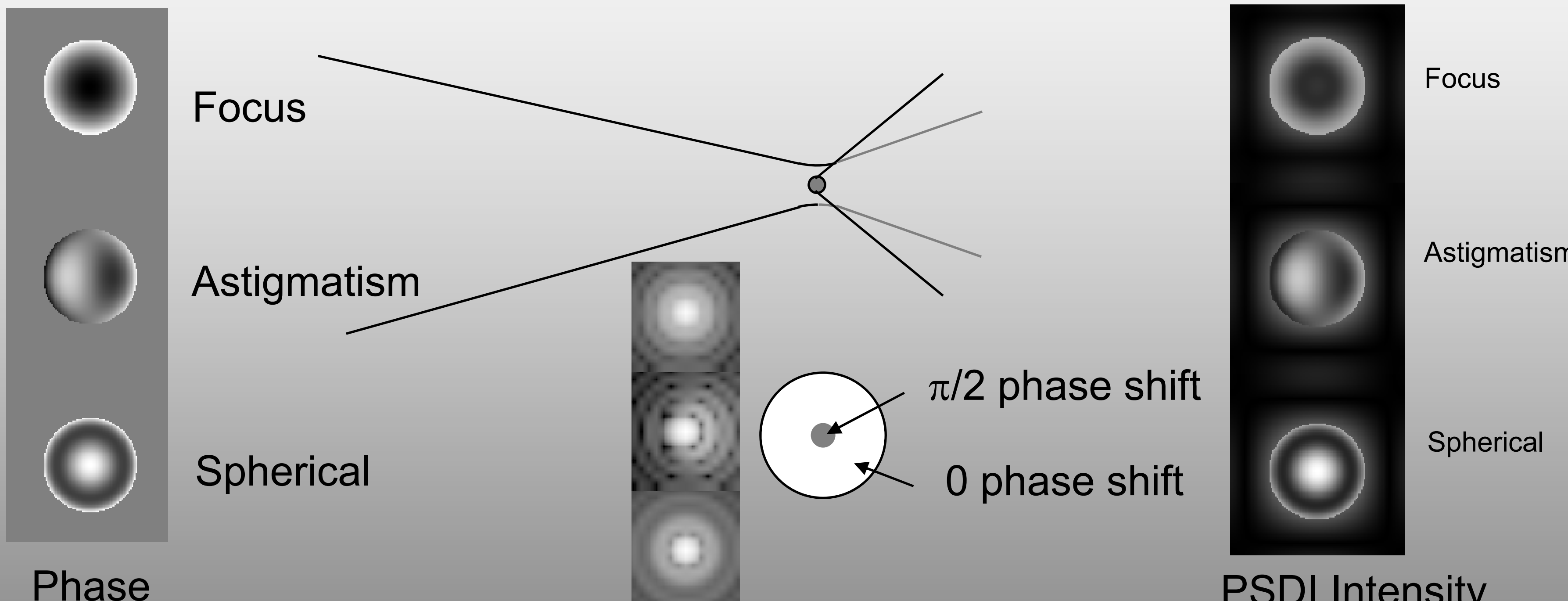
Example: Phase Shifting Point Diffraction Interferometer





- Focal Plane Pinhole and ¼ wave phase shift at center of beam approximately 1/2 diffraction limit

Example: Phase Shifting Point Diffraction Interferometer

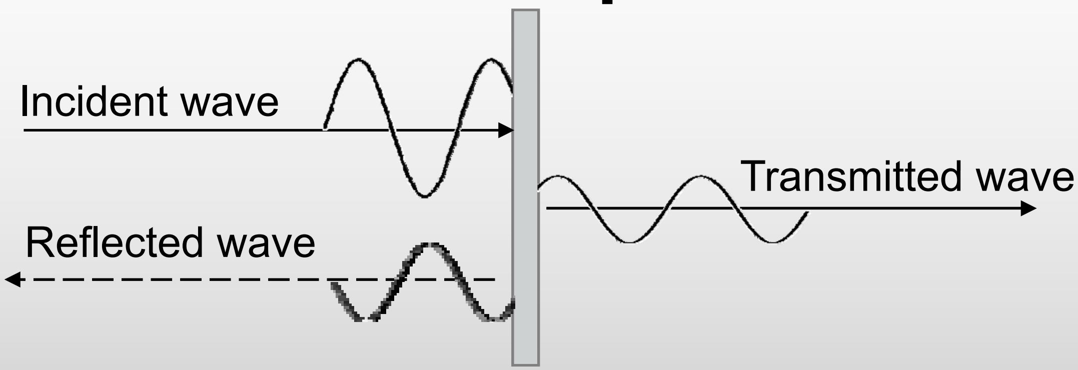


Phase Aberration

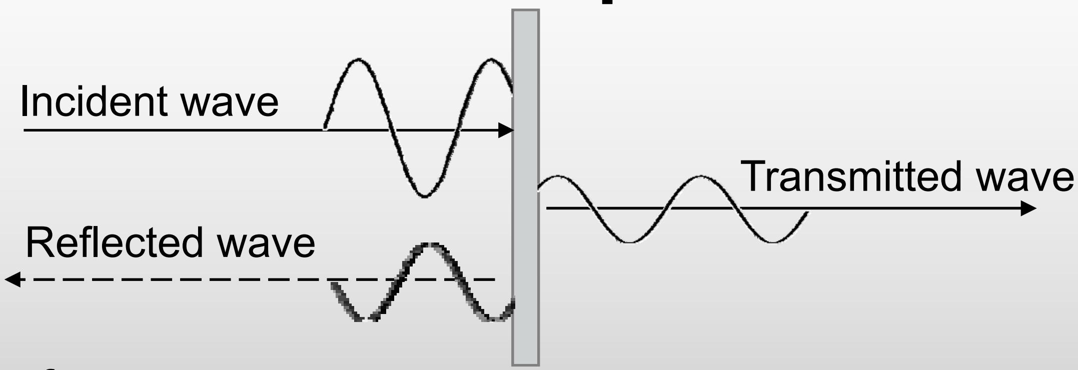
PSDI Intensity measurements

- Focal Plane Pinhole and ¼ wave phase shift at center of beam approximately 1/2 diffraction limit

Beam Splitter



Beam Splitter



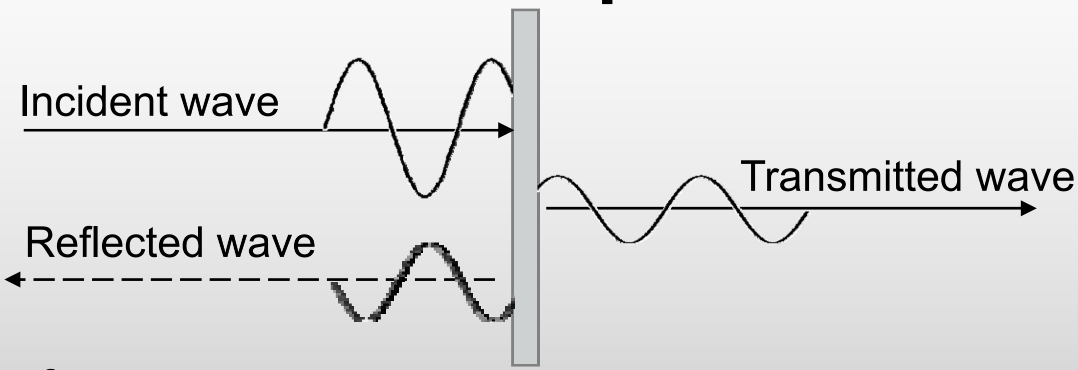
At the interface:

- EM wave magnitudes must match $E_i + E_r = E_t$
- Energy must balance $|E_i|^2 = |E_r|^2 + |E_t|^2$

E complex, -> 3 equations in 4 unknowns

Free variable is Transmissivity of the beam splitter $|E_t|^2 = T|E_i|^2$

Beam Splitter



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E complex, -> 3 equations in 4 unknowns

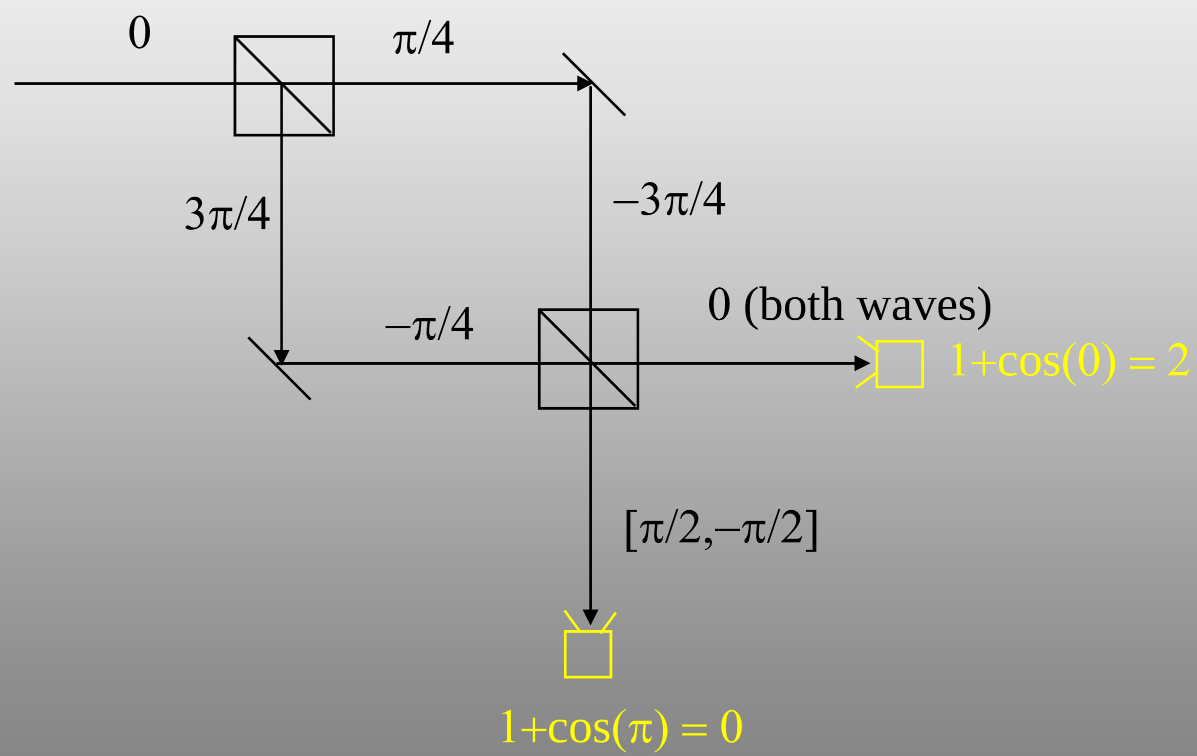
Free variable is Transmissivity of the beam splitter $|E_t|^2 = T|E_i|^2$

a little math, and… $\frac{|E_t|}{|E_i|} = \cos(\phi_t) \quad \frac{|E_r|}{|E_i|} = -\cos(\phi_r) \quad \Rightarrow \quad \phi_t - \phi_r = -\pi/2$

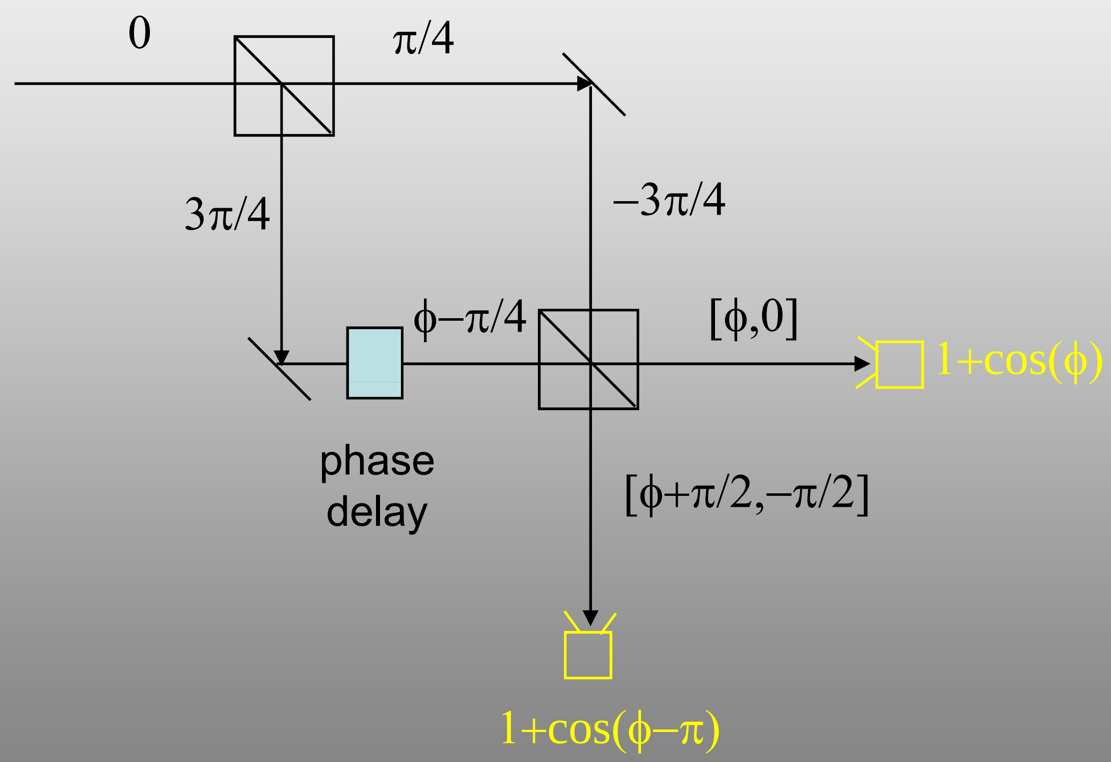
The transmitted and reflected waves differ by $\pi/2$ (90°)

- Special case: 50/50 beamsplitter $T = 0.5 \quad \phi_t = \pi/4 \quad \phi_r = 3\pi/4$
- Special case: mirror $T = 0 \quad \phi_t = \pi/2 \quad \phi_r = \pi$

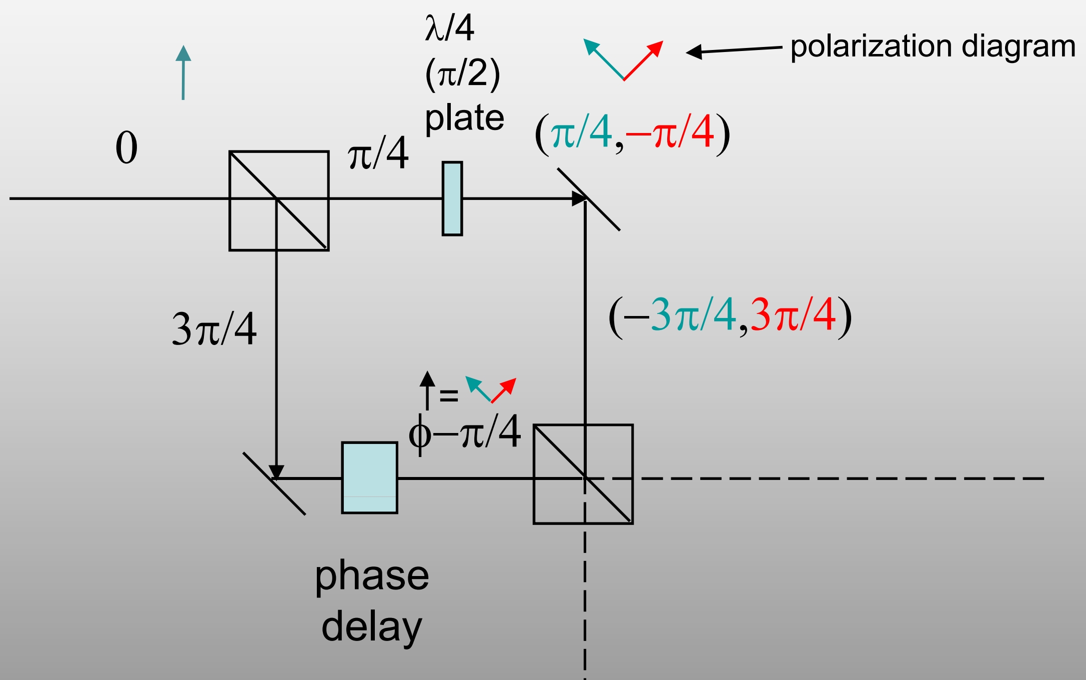
Beam Combiner (Interferometer)



Interferometeric Testing



Quadrature Phase Interferometer



Quadrature Phase Interferometer

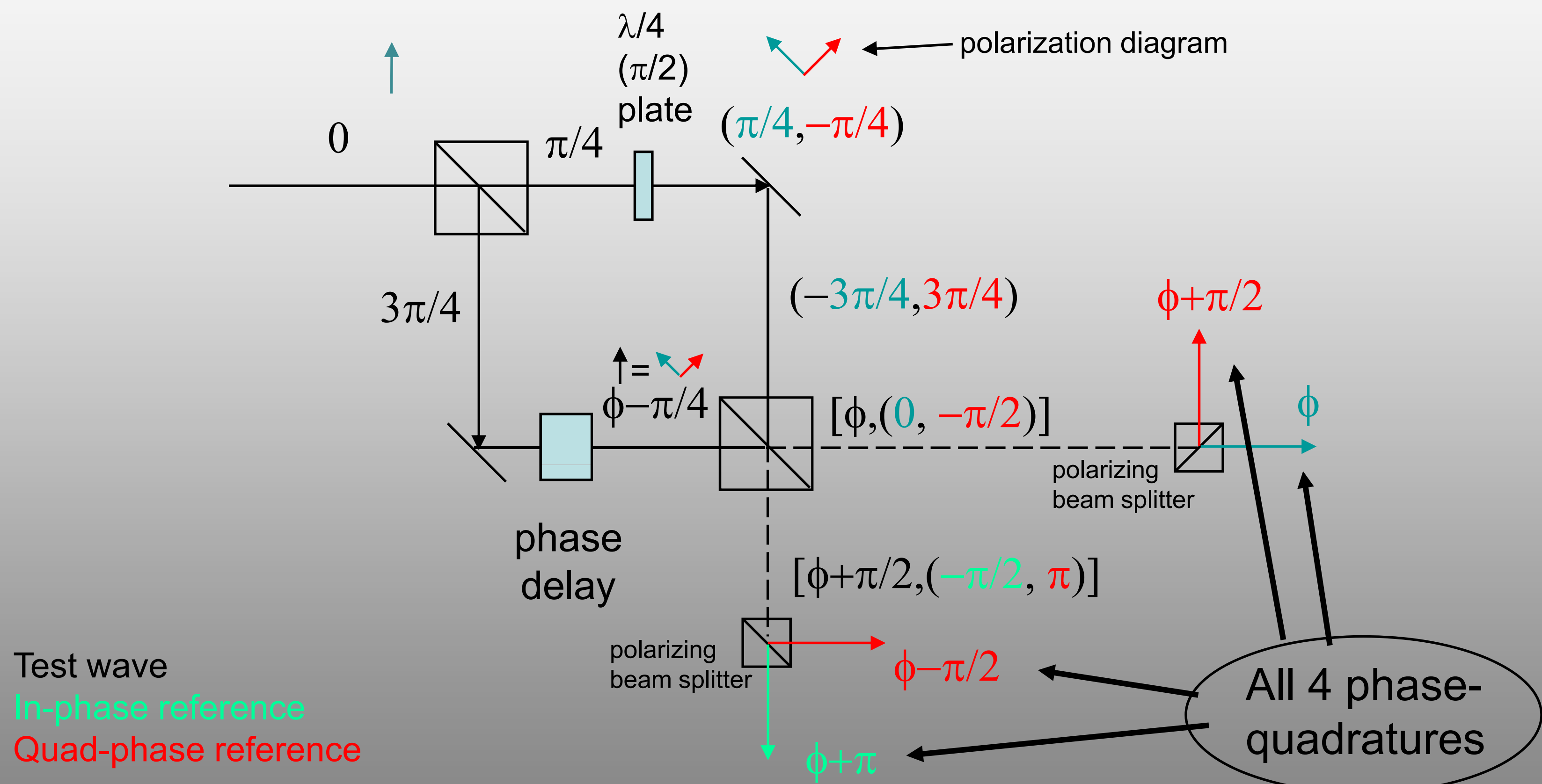
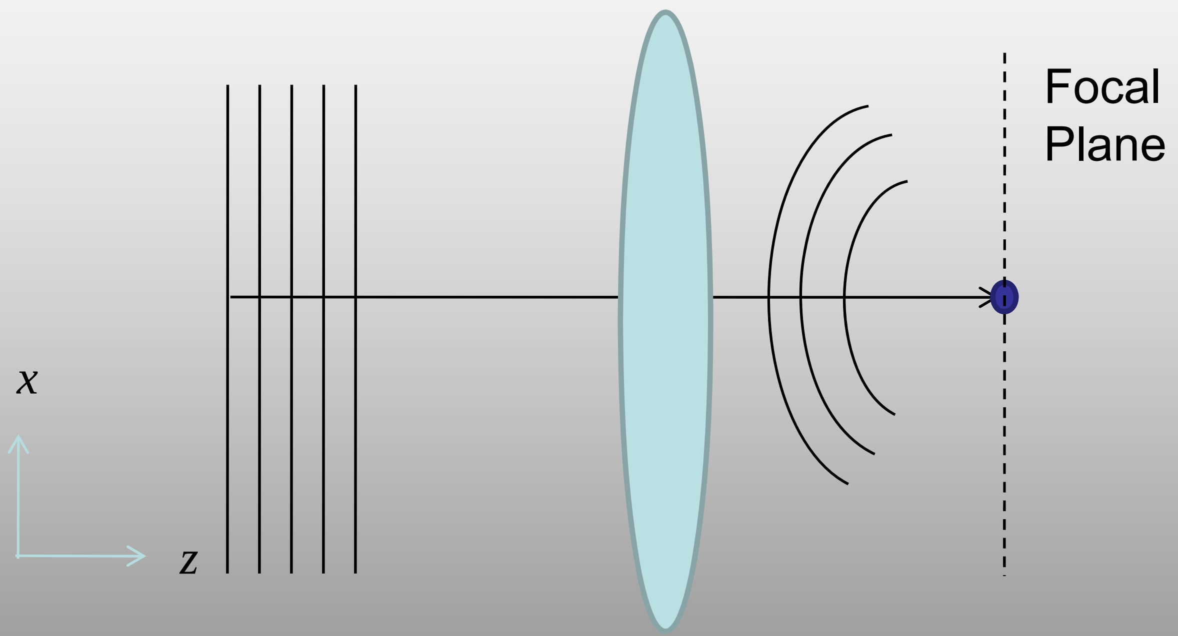


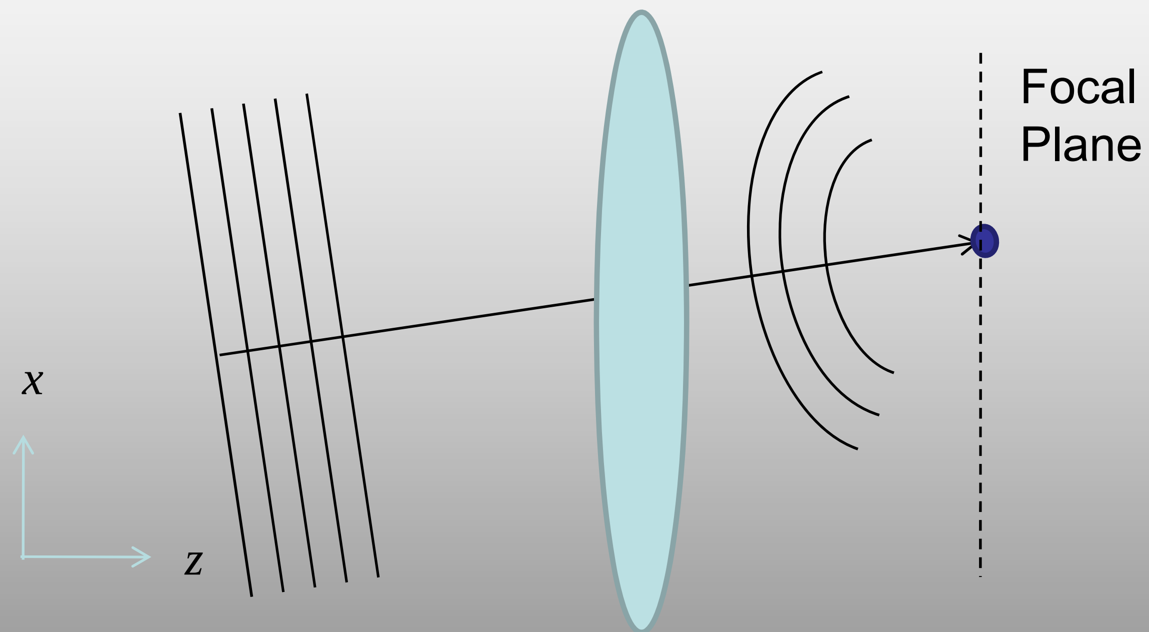
Image Formation



$Ae^{i2\pi z/\lambda}$

$A\delta(x)$

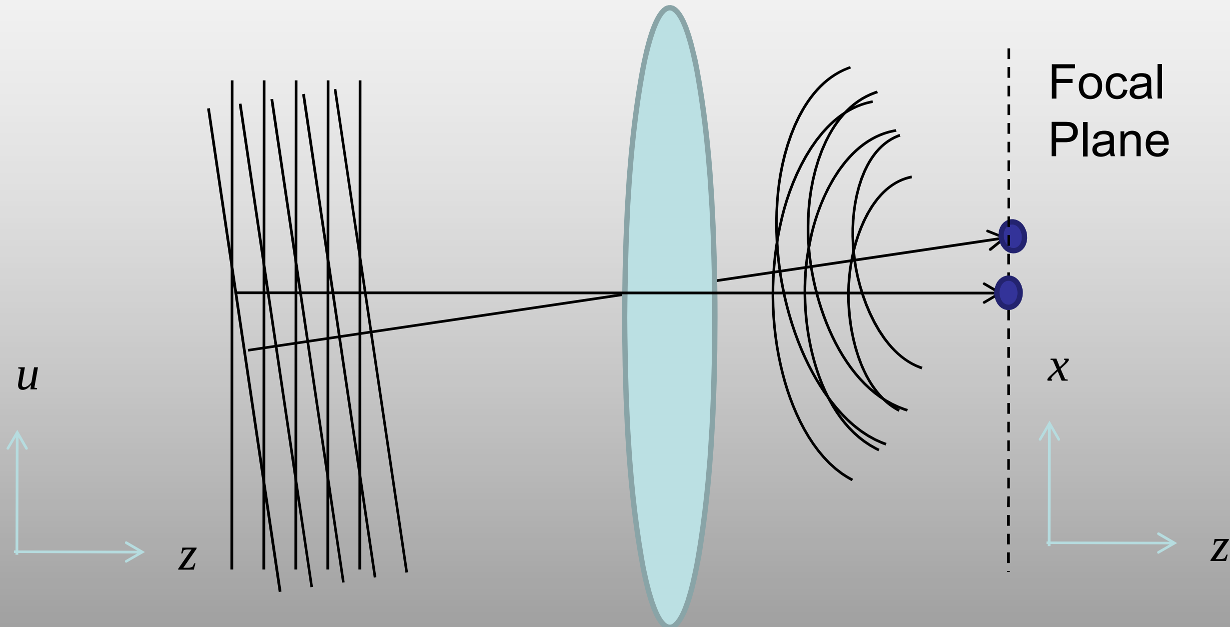
Image Formation



$A e^{i 2\pi (z - \theta x)/\lambda}$

$A \delta(x + f\theta)$

Image Formation

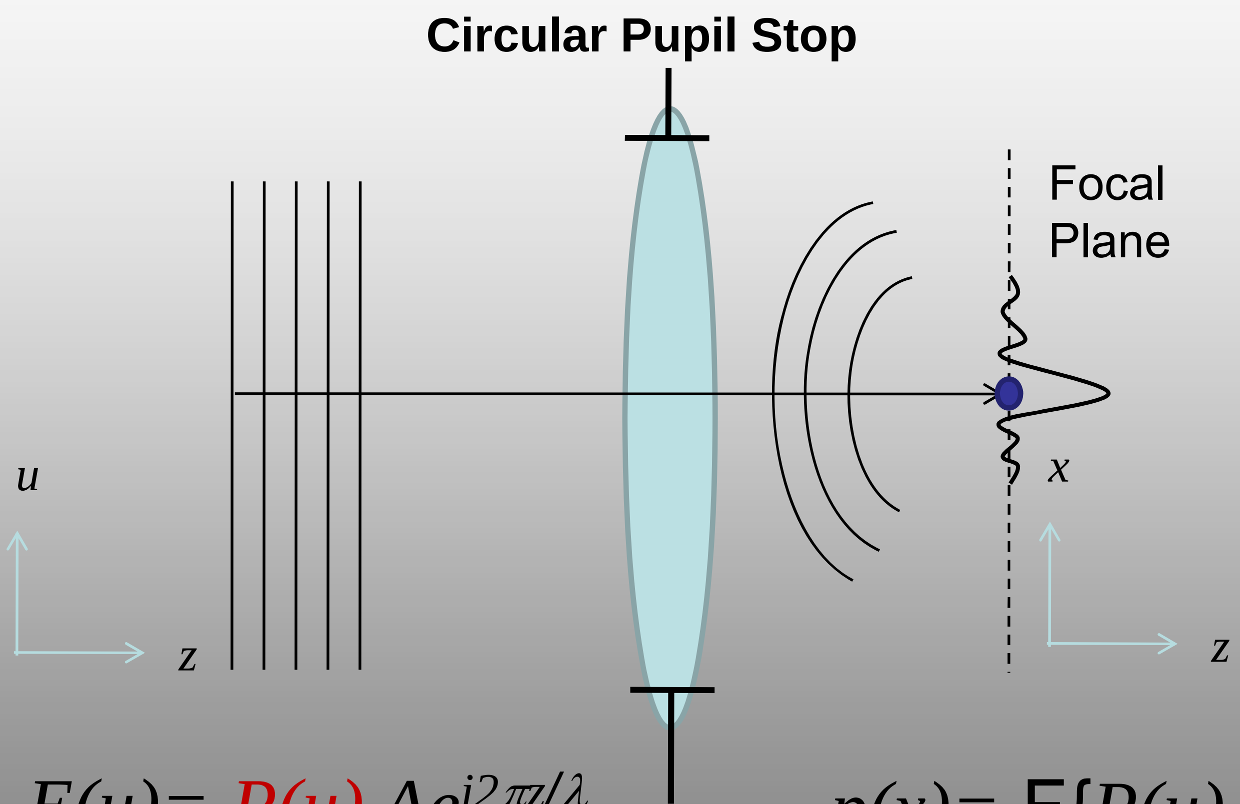


$$E(u) = \Sigma A(\theta) e^{i2\pi(z-\theta x)/\lambda}$$

$$F(x) = \Sigma A(\theta)\,\delta(x + f\theta)$$
$$= A(x/f)$$

$\therefore$ $E(u)$ and $F(x)$ are Fourier Transform Pairs

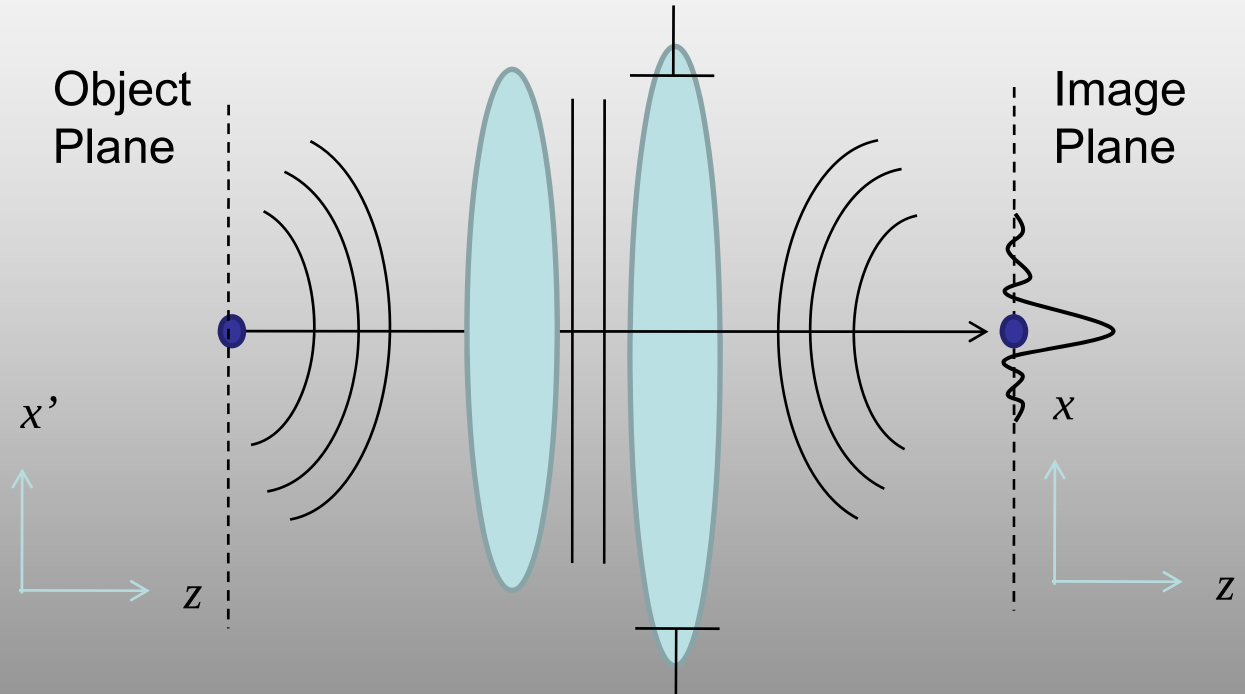
Image Formation



$$E(u) = P(u)\, Ae^{i2\pi z/\lambda}$$

$$p(x) = \mathcal{F}\{P(u)\}$$

Image Formation

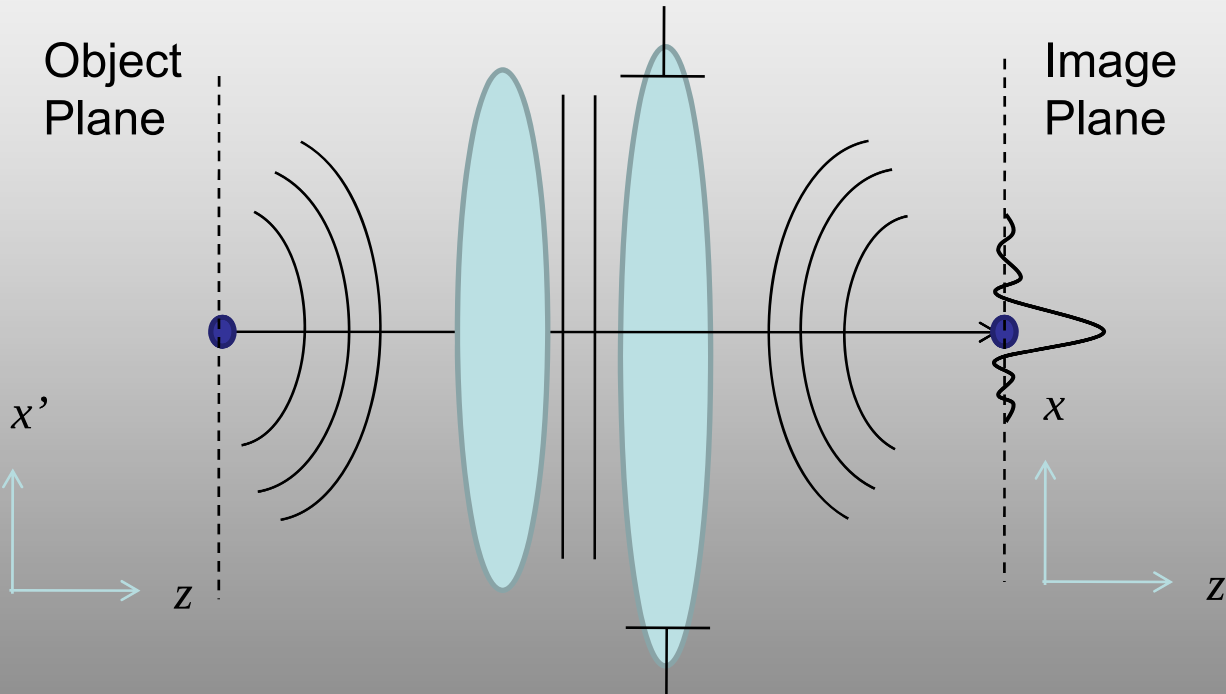


$\delta(x')$ ⟶ Plane Wave ⟶ $p(x)$

$o(x')$ ⟶ $A(u)$ = F{$o(x')$}$P(u)$ ⟶ $i(x)=p(x) \otimes o(x)$

Point Spread Function (PSF)

The Point Spread Function is the distribution of energy in the image plane in response to a point source in the object plane

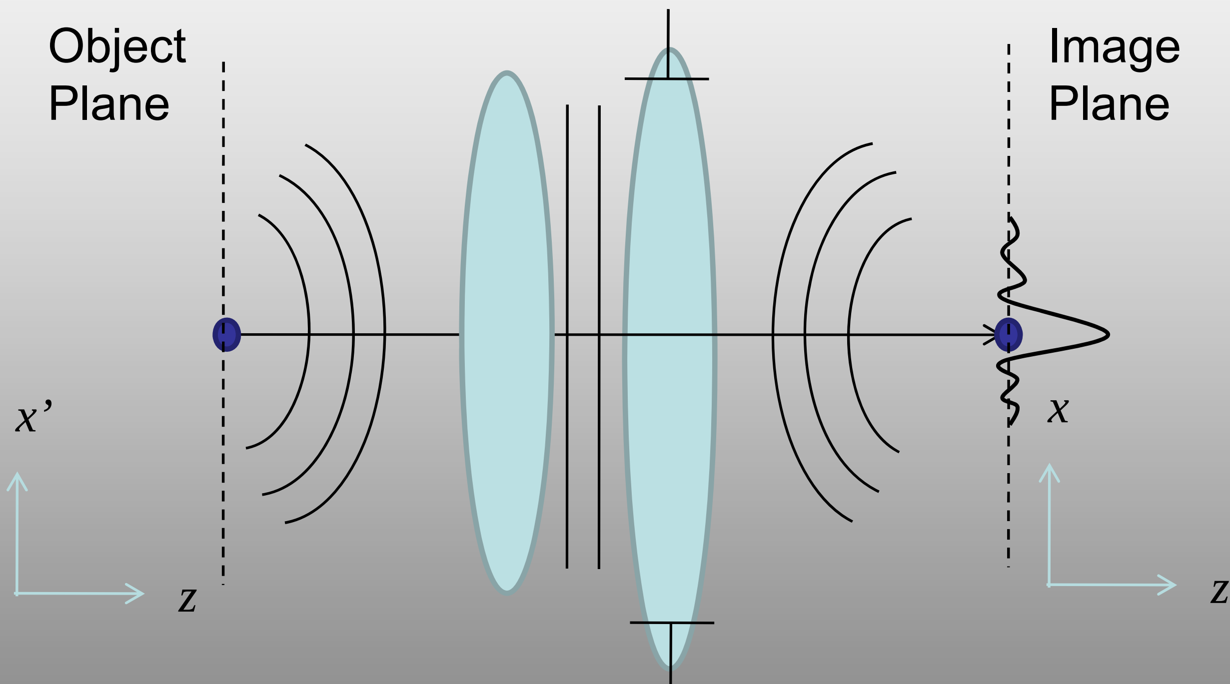


$\delta(x')$

$PSF(x) = |p(x)|^2$

Optical Transfer Function (OTF)

The Optical Transfer Function is the Fourier transform of the Point Spread Function



$\delta(x')$ $\quad OTF(u) = \mathrm{F}\{PSF(x)\}$ $\quad PSF(x)=|p(x)|^2$

$$= \int A(u-u')\,A(u')\,du'$$

Optical and Modulation Transfer Functions

- Optical transfer function (OTF) – how a sinusoidal intensity pattern in the object plane is imaged in the focal plane
  - Modulation transfer function (MTF) is the amplitude part,

    MTF = |OTF|
  - Phase transfer function (PTF) is the phase part,

    PTF = arg(OTF)
- *Assuming an object made up of incoherent point sources*, each point in the object plane is blurred in the image plane by the point-spread-function (PSF), which is the Fourier transform of the OTF

$$|i(x)|^2 = PSF(x) \otimes |o(x)|^2$$
$$I(f) = OTF(f)\, O(f)$$

Example MTFs

with varying amounts of aberration in the optical system